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2008 J. Phys. A: Math. Theor. 41 065207

(http://iopscience.iop.org/1751-8121/41/6/065207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 065207 (21pp) doi:10.1088/1751-8113/41/6/065207

A rigorous analysis of high-order electromagnetic invisibility cloaks*

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Received 4 November 2007, in final form 20 December 2007
Published 29 January 2008
Online at stacks.iop.org/JPhysA/41/065207

Abstract
There is currently a great deal of interest in the invisibility cloaks recently proposed by Pendry *et al* that are based on the transformation approach. They obtained their results using first-order transformations. In recent papers, Hendi *et al* and Cai *et al* considered invisibility cloaks with high-order transformations. In this paper, we study high-order electromagnetic invisibility cloaks in transformation media obtained by high-order transformations from general anisotropic media. We consider the case where there is a finite number of spherical cloaks located in different points in space. We prove that for any incident plane wave, at any frequency, the scattered wave is identically zero. We also consider the scattering of finite-energy wave packets. We prove that the scattering matrix is the identity, i.e., that for any incoming wave packet the outgoing wave packet is the same as the incoming one. This proves that the invisibility cloaks cannot be detected in any scattering experiment with electromagnetic waves in high-order transformation media, and in particular in the first-order transformation media of Pendry *et al*. We also prove that the high-order invisibility cloaks, as well as the first-order ones, cloak passive and active devices. The cloaked objects completely decouple from the exterior. Actually, the cloaking outside is independent of what is inside the cloaked objects. The electromagnetic waves inside the cloaked objects cannot leave the concealed regions and vice versa, the electromagnetic waves outside the cloaked objects cannot go inside the concealed regions. As we prove our results for media that are obtained by transformation from general anisotropic materials, we prove that it is possible to cloak objects inside general crystals.

PACS numbers: 41.20.Jb, 02.30.Tb, 02.30.Zz, 02.60.Lj

* Research partially supported by CONACYT under Project P42553-F.

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Mathematics Subject Classification: 35L45, 35L50, 35L80, 35P25, 35Q60, 78A25, 78A45

1. Introduction

Recently [1] considered electromagnetic invisibility cloaks based on the transformation method that offer the theoretical and practical possibility of hiding objects from observation by electromagnetic waves. The results in [1] were obtained in the geometrical optics approximation. Numerical simulations were reported in [2, 3] and an experimental verification of cloaking was given in [4]. In the case of one spherically symmetric cloak [5] proved by an explicit calculation in spherical coordinates that the scattered wave is identically zero at each frequency and that the electromagnetic waves outside the spherical cloaked object cannot enter the concealed region. These papers considered first-order transformations. The recent papers [6, 7] consider high-order transformations that offer new possibilities. In all these studies the transformation media were obtained by transformations from isotropic media.

In this paper, we consider electromagnetic invisibility cloaks in high-order transformation media that are obtained by high-order transformations from general anisotropic media. Moreover, we assume that there are several cloaked objects located in different points in space.

We prove that for any incident plane wave, at any frequency, the scattered wave is identically zero. This generalizes, with a different proof, the results obtained in [5]. Note that in our case separation of variables cannot be used as there is no symmetry since we have a finite number of cloaks in different points in space and also because we transform from general anisotropic media.

We also consider the scattering of finite-energy wave packets. We prove that the scattering matrix is the identity, i.e., that for any incoming wave packet the outgoing wave packet is the same as the incoming one.

These results prove that the invisibility cloaks cannot be detected in any scattering experiment with electromagnetic waves in high-order transformation media, and in particular, in the first-order transformation media of [1].

We also prove that the high-order invisibility cloaks, as well as the first-order ones, cloak passive and active devices. The cloaked objects completely decouple from the exterior. Actually, the cloaking outside is independent of what is inside the cloaked objects. The electromagnetic waves inside the cloaked objects cannot leave the concealed regions and vice versa, the electromagnetic waves outside the cloaked objects cannot go inside the concealed regions.

In [8], we considered first-order transformation media. Here, we present the generalization to high-order transformations, and we add new results.

The fact that the electromagnetic waves inside the cloaked objects cannot leave the concealed region, which we had already proved in [8] in the case of first-order transformations, has recently been verified in the case of one spherical cloak in a first-order transformation medium, by an explicit computation in spherical coordinates, by [9], where also a physical mechanism based in surface voltages is presented to physically explain why the electromagnetic waves cannot leave the concealed region.

Our results are based in Von Neumann's method of self-adjoint extensions. This is a very powerful technique that allows us to settle in an unambiguous way the mathematical problems posed by the singularities of the inverse of the permittivity and the permeability of the transformation media in the boundary of the cloaked objects. It also allows us to identify the appropriate boundary condition when cloaking is formulated as a boundary value problem. Namely, that the tangential components of the electric and the magnetic fields have to vanish at the outside of the boundary of the cloaked objects (see remark 2.4). We have already proven this result for first-order transformations in [8]. This boundary condition is self-adjoint in our case because the permittivity and the permeability are degenerate at the boundary of the cloaked objects.

As we prove our results for media that are obtained by transformation from general anisotropic materials, we prove that it is possible to cloak objects inside general crystals.

As it is often the case in the papers on electromagnetic invisibility cloaks, I make the assumption that the media are not dispersive. This is a widely used idealization. As is well known, metamaterials are dispersive, and, furthermore, when the permittivity and the permeability have eigenvalues less than one, dispersion comes into play in order that the group velocity does not exceeds the speed of light. This idealization means that we have to take a narrow enough range of frequencies in order that we can analyze the cloaking effect without taking dispersion into account. In practice, this means that cloaking will only be approximate.

For a related method for cloaking in two dimensions see [10]. For earlier results in cloaking for conductivity problems see [11], the references quoted there, and [12, 13]. In [14], cloaking is studied in the context of the Dirichlet to Neumann map. For a cylindrical invisibility cloak with first-order transformation see [15]. See also [16, 17] for other related results

The paper is organized as follows. In section 2, we introduce our formalism, we give the definition of solutions with locally finite energy and we obtain the cloaking boundary condition. Moreover, we prove that the electromagnetic waves inside the cloaked objects cannot go outside and vice versa. In section 3, we prove that the scattered waves are zero for all frequencies and all incoming plane waves. In section 4, we prove that the scattering matrix is the identity for all incoming wave packets. In section 5, we give the proof of theorems 2.3 and 2.5. In section 6, we discuss generalizations of our results. Finally, we give a brief conclusion and outlook where we also comment on cloaking objects inside general anisotropic media, in particular inside general crystals.

2. Electromagnetic cloaking

Let us consider Maxwell's equations in $\mathbb{R}^3$, in the time domain,

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \qquad \nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D}, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = 0, \tag{2.2}$$

and in the frequency domain, assuming a periodic time dependence of $\mathbf{E}, \mathbf{H}$ given by $\mathrm{e}^{-\mathrm{i}\omega t}$, with ω the frequency,

$$\nabla \times \mathbf{E} = \mathrm{i}\omega\mathbf{B}, \qquad \nabla \times \mathbf{H} = -\mathrm{i}\omega\mathbf{D}, \qquad \omega \neq 0, \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = 0, \tag{2.4}$$

where we have suppressed the factor $\mathrm{e}^{-\mathrm{i}\omega t}$ in both sides. Note that (2.4) follows from (2.3).

We study the propagation of electromagnetic waves—that satisfy Maxwell's equation—in the case where there is a finite number of high-order spherical invisibility cloaks located in different points in space.

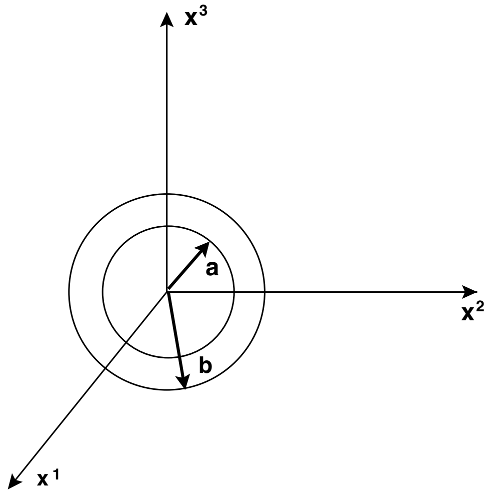

Figure 1. One spherical cloak centered at zero.

For simplicity let us first consider the case where there is only one cloak located at $\mathbf{x} = 0$ (see figure 1). We designate the Cartesian coordinates of $\mathbf{x}$ by x^λ, $\lambda = 1, 2, 3$. To define the high-order transformation media we introduce another copy of $\mathbb{R}^3$, denoted by $\mathbb{R}^3_0$. The points in $\mathbb{R}^3_0$ are denoted by $\mathbf{y}$ with coordinates y^λ, $\lambda = 1, 2, 3$. We designate by $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, $\hat{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|$ unit vectors. Consider the following transformation from $\mathbb{R}^3_0\backslash\{0\}$ to $\mathbb{R}^3$ [1, 7]:

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := g(|\mathbf{y}|)\hat{\mathbf{y}}. \tag{2.5}$$

In spherical coordinates this transformation changes the radial coordinate but leaves the angular coordinates constant, i.e., $|\mathbf{x}| = g(|\mathbf{y}|)$, $\hat{\mathbf{x}} = \hat{\mathbf{y}}$. Given $0 < a < b$ we wish that this transformation sends the punctuated ball $0 < |\mathbf{y}| \leqslant b$ onto the concentric shell $a < |\mathbf{x}| \leqslant b$, that it is the identity for $|\mathbf{y}| \geqslant b$ and that it is one-to-one. Then, we assume that g satisfies the following conditions.

Definition 2.1. *For any positive numbers a, b with $0 < a < b$, we say the g is a cloaking function in $[0, b]$ if $g(\rho)$ is twice continuously differentiable on $[0, b]$, $g(0) = a$, $g(b) = b$, and $g'(\rho) := \frac{\mathrm{d}}{\mathrm{d}\rho} g(\rho) > 0$, $\rho \in [0, b]$.*

These high-order transformations were introduced in [7] for the case of a cylindrical cloak. They imposed the condition $g'(b) = 1$. Since for our spherical cloaks we do not need this condition we do not assume it. We define

$$\begin{aligned} &\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := g(|\mathbf{y}|)\hat{\mathbf{y}}, \qquad \text{for} \qquad 0 < |\mathbf{y}| \leqslant b, \\ &\mathbf{x} = \mathbf{x}(\mathbf{y}) := \mathbf{y}, \qquad \text{for} \qquad |\mathbf{y}| \geqslant b. \end{aligned} \tag{2.6}$$

With these conditions (2.6) is a bijection from $\mathbb{R}^3_0\backslash\{0\}$ onto $\mathbb{R}^3\backslash B_a(0)$, where by

$$B_r(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^3 : |x - \mathbf{x}_0| \leqslant r\} \tag{2.7}$$

we denote the closed ball of center $\mathbf{x}_0$ and radius r. Moreover, it blows up the point 0 onto the sphere $|\mathbf{x}| = a$. It sends the punctuated ball $0 < |\mathbf{y}| \leqslant b$ onto the concentric shell $a < |\mathbf{x}| \leqslant b$ and it is the identity for $|\mathbf{y}| \geqslant b$. It is twice continuously differentiable away from the sphere $|\mathbf{y}| = b$, where it can have discontinuities in the derivatives depending on the values of the derivatives of g at b.

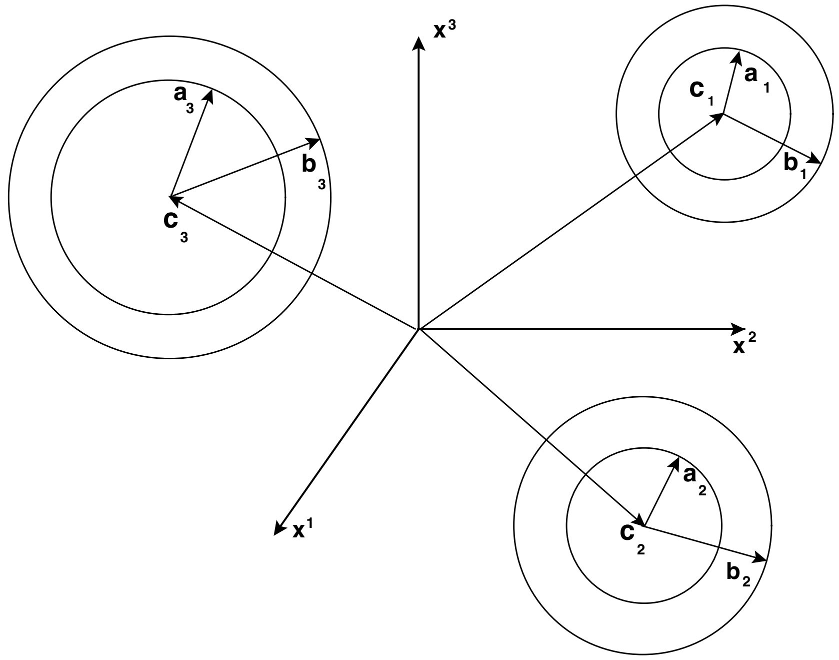

Figure 2. Three spherical cloaks centered at $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$.

In [7], the quadratic case

$$g(\rho) = \left[1 - \frac{a}{b} + p(\rho - b)\right]\rho + a$$

with $p \in \mathbb{R}$ was discussed in connection with a cylindrical cloak in an approximate transformation medium. In [1] the first-order case $g(\rho) = \frac{b-a}{b}\rho + a$ was considered.

The closed ball $K := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leqslant a\}$ is the region that we wish to conceal, and we call it the cloaked object. The spherical shell $a < |\mathbf{x}| \leqslant b$ is the cloaking layer. The union of the cloaked object and the cloaking layer is the spherical cloak, that in this case is just the closed ball of center zero and radius b. The domain $|\mathbf{x}| > b$ is the exterior of the spherical cloak.

To have several cloaks we just put a finite number of these spherical cloaks in different points in space at a finite positive distance from each other, in order that they do not intersect (see figure 2). Let us take as centers of the cloaks points $\mathbf{c}_j \in \mathbb{R}^3$, $j = 1, 2, \ldots, N$ where N is the number of cloaks and $\mathbf{c}_j \neq \mathbf{c}_l, j \neq l, 1 \leqslant j, l \leqslant N$. We take $0 < a_j < b_j$, and cloaking functions g_j that satisfy the conditions of definition 2.1 for $a_j, b_j, j = 1, 2, 3, \ldots, N$, and we define the following transformation from $\mathbb{R}^3_0 \setminus \{\mathbf{c}_1, \mathbf{c}_2, \ldots, c_N\}$ to $\mathbb{R}^3$:

$$\begin{aligned} &\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := \mathbf{c}_j + g_j(|\mathbf{y} - \mathbf{c}_j|)\widehat{\mathbf{y} - \mathbf{c}_j}, \qquad \mathbf{y} \in B_{b_j}(\mathbf{c}_j), \quad j = 1, 2, \ldots, N, \\ &\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := \mathbf{y}, \qquad \mathbf{y} \in \mathbb{R}^3_0 \Big\backslash \cup_{j=1}^N B_{b_j}(\mathbf{c}_j), \end{aligned} \tag{2.8}$$

where $B_{b_j}(\mathbf{c}_j)$ are balls in $\mathbb{R}^3_0$.

The cloaked objects that we wish to conceal are given by

$$K_j := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{c}_j| \leqslant a_j\}, \qquad j = 1, 2, \ldots, N. \tag{2.9}$$

The concentric spherical shells $a_j < |\mathbf{x} - \mathbf{c}_j| \leqslant b_j$, $j = 1, 2, \ldots, N$ are the cloaking layers. The spherical cloaks are the balls $B_{b_j}(\mathbf{c}_j)$ in $\mathbb{R}^3$. We denote by K the union of all the cloaked objects,

$$K := \cup_{j=1}^{N} K_j. \tag{2.10}$$

The domain

$$\mathbb{R}^3 \Big\backslash \cup_{j=1}^{N} B_{b_j}(\mathbf{c}_j) \tag{2.11}$$

is the exterior of the all the spherical cloaks. We assume that the spherical cloaks are at a positive distance of each other,

$$\text{min distance } \big(B_{b_j}(\mathbf{c}_j), B_{b_l}(\mathbf{c}_l)\big) > 0, \qquad j \neq l, \quad j, l = 1, 2, \ldots, N.$$

Denote

$$\Omega_0 := \mathbb{R}_0^3 \Big\backslash \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_N\}, \qquad \Omega := \mathbb{R}^3 \backslash K.$$

Then, (2.8) is a bijection from Ω_0 onto Ω, and for $j = 1, 2, \ldots, N$ it blows up the point $\mathbf{c}_j$ onto the sphere $|\mathbf{x} - \mathbf{c}_j| = a_j$. It sends the punctuated ball $0 < |\mathbf{y} - \mathbf{c}_j| \leqslant b_j$ onto the concentric shell $a_j < |\mathbf{x} - \mathbf{c}_j| \leqslant b_j$ and it is the identity for $\mathbf{y} \in \mathbb{R}_0^3 \Big\backslash \text{interior}\,\big(\cup_{j=1}^{N} B_{b_j}(\mathbf{c}_j)\big)$. It is twice continuously differentiable away from the spheres $|\mathbf{y} - \mathbf{c}_j| = b_j$, where it can have discontinuities in the derivatives depending on the values of the derivatives of g_j at b_j.

For any open set O and for any $n = 1, 2, \ldots,$ let us denote by $C^n(O)$ the set of all $\mathbb{C}$-valued functions that are continuous together with all its derivatives of order up to n, and by $C_0^n(O)$ the functions in $C^n(O)$ that have compact support in O, i.e. such that the closure of the set of points where they are different from zero is bounded in O. In other words, the closure of set of points where the functions are different from zero is bounded, and they are zero in a neighborhood of the boundary of O. By $C(O)$, $C_0(O)$, we denote, respectively, the continuous functions in O and the continuous functions with compact support in O.

We denote the elements of the Jacobian matrix by $A_{\lambda'}^{\lambda}$,

$$A_{\lambda'}^{\lambda} := \frac{\partial x^{\lambda}}{\partial y^{\lambda'}}. \tag{2.12}$$

Note that $A_{\lambda'}^{\lambda} \in \mathbf{C}^1\big(\Omega_0 \Big\backslash \cup_{j=1}^{N} \partial B_{b_j}(\mathbf{c}_j)\big)$, and that it can have discontinuities on $\cup_{j=1}^{N} \partial B_{b_j}(\mathbf{c}_j)$ depending on the derivatives of g_j at b_j. We designate by $A_{\lambda}^{\lambda'}$ the elements of the Jacobian of the inverse bijection, $\mathbf{y} = \mathbf{y}(\mathbf{x}) = f^{-1}(\mathbf{x})$,

$$A_{\lambda}^{\lambda'} := \frac{\partial y^{\lambda'}}{\partial x^{\lambda}}. \tag{2.13}$$

$A_{\lambda}^{\lambda'} \in \mathbf{C}^1\big(\Omega \Big\backslash \cup_{j=1}^{N} \partial B_{b_j}(\mathbf{c}_j)\big)$, and it can have discontinuities on $\cup_{j=1}^{N} \partial B_{b_j}(\mathbf{c}_j)$ depending on the derivatives of g_j at b_j. Let us denote by Δ the determinant of the Jacobian matrix (2.12). See (5.2) for the calculation of Δ in closed form. Note that it is infinite at $\partial K = \partial \Omega$.

We take here the *material interpretation* and we consider our transformation as a bijection between two different spaces, Ω_0 and Ω. However, our transformation can be considered, as well, as a change of coordinates in Ω_0. Of course, these two points of view are mathematically equivalent. This means, in particular, that under our transformation Maxwell's equations in Ω_0 and in Ω have the same invariance that they have under change of coordinates in 3-space (see, for example, [18]). Let us denote by $\mathbf{E}_0, \mathbf{H}_0, \mathbf{B}_0, \mathbf{D}_0, \varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, respectively, the electric and magnetic fields, the magnetic induction, the electric displacement, and the permittivity and permeability of Ω_0. $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are positive, Hermitian matrices that are constant in Ω_0.

The electric field is a covariant vector that transforms as

$$E_{\lambda}(\mathbf{x}) = A_{\lambda}^{\lambda'}(\mathbf{y}) E_{0,\lambda'}(\mathbf{y}). \tag{2.14}$$

The magnetic field $\mathbf{H}$ is a covariant pseudo-vector, but as we only consider space transformations with positive determinant, it also transforms as in (2.14). The magnetic induction $\mathbf{B}$ and the electric displacement $\mathbf{D}$ are contravariant vector densities of weight one that transform as

$$B^{\lambda}(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A^{\lambda}_{\lambda'}(\mathbf{y}) B_0^{\lambda'}(\mathbf{y}), \tag{2.15}$$

with the same transformation for $\mathbf{D}$. The permittivity and permeability are contravariant tensor densities of weight one that transform as

$$\varepsilon^{\lambda\nu}(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A^{\lambda}_{\lambda'}(\mathbf{y}) A^{\nu}_{\nu'}(\mathbf{y}) \varepsilon_0^{\lambda'\nu'}(\mathbf{y}), \tag{2.16}$$

with the same transformation for $\mu^{\lambda\nu}$. Maxwell's equations (2.1)–(2.4) are the same in both spaces Ω and Ω_0. Let us denote by $\varepsilon_{\lambda\nu}, \mu_{\lambda\nu}, \varepsilon_{0\lambda\nu}, \mu_{0\lambda\nu}$, respectively, the inverses of the corresponding permittivity and permeability. They are covariant tensor densities of weight minus one that transform as

$$\varepsilon_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A^{\lambda'}_{\lambda}(\mathbf{y}) A^{\nu'}_{\nu}(\mathbf{y}) \varepsilon_{0\lambda'\nu'}(\mathbf{y}), \qquad \mu_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A^{\lambda'}_{\lambda}(\mathbf{y}) A^{\nu'}_{\nu}(\mathbf{y}) \mu_{0\lambda'\nu'}(\mathbf{y}). \tag{2.17}$$

Note that

$$\det \varepsilon^{\lambda\nu} = \Delta^{-1} \det \varepsilon_0^{\lambda\nu}, \qquad \det \mu^{\lambda\nu} = \Delta^{-1} \det \mu_0^{\lambda\nu}, \tag{2.18}$$

$$\det \varepsilon_{\lambda\nu} = \Delta \det \varepsilon_{0\lambda\nu}, \qquad \det \mu_{\lambda\nu} = \Delta \det \mu_{0\lambda\nu}. \tag{2.19}$$

Then, by (2.18), (2.19) and (5.2) the matrices $\varepsilon^{\lambda\nu}, \mu^{\lambda\nu}$ are degenerate at ∂K and the matrices $\varepsilon_{\lambda\nu}, \mu_{\lambda\nu}$ are singular at ∂K.

We face now the problem that as $\varepsilon^{\lambda\nu}$ and $\mu^{\lambda\nu}$ are degenerate at the boundary of the cloaked objects K we have to make precise what do we mean by a solution to Maxwell's equations. As we will see the standard rules that we apply in non-degenerate situations do not apply here. These type of problems are not unusual in mathematical physics and there is a well-established method to deal with them. Namely, Von Neumann's theory of self-adjoint extensions [19, 20]. Let us first consider the problem in Ω. We write Maxwell's equations in Schrödinger form. For this purpose we denote by $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$, respectively, the matrices with entries $\varepsilon_{\lambda\nu}$ and $\mu_{\lambda\nu}$. Recall that $(\nabla \times \mathbf{E})^{\lambda} = s^{\lambda\nu\rho} \frac{\partial}{\partial x_\nu} E_\rho$, where $s^{\lambda\nu\rho}$ is the permutation contravariant pseudo-density of weight -1 (see section 6 of chapter II of [18], where a different notation is used).

We define the following formal differential operator:

$$a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \mathrm{i} \begin{pmatrix} \boldsymbol{\varepsilon} \nabla \times \mathbf{H} \\ -\boldsymbol{\mu} \nabla \times \mathbf{E} \end{pmatrix}. \tag{2.20}$$

Here, as usual, we denote $\boldsymbol{\varepsilon} \nabla \times \mathbf{H} := \varepsilon_{\lambda\nu} (\nabla \times \mathbf{H})^{\nu}$ and $\boldsymbol{\mu} \nabla \times \mathbf{E} = \mu_{\lambda\nu} (\nabla \times \mathbf{E})^{\nu}$.

Equation (2.1) is equivalent to

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{2.21}$$

and equation (2.3) is equivalent to

$$\omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{2.22}$$

Note that since the matrices ϵ, μ are singular at $\partial\Omega$, the operator a_Ω has coefficients that are singular at $\partial\Omega$. This is the reason why we have to be careful when defining the solutions.

We have to define equation (2.20) in an appropriate linear subspace of the Hilbert space of all finite-energy fields in Ω, that we define now. We designate by $\mathcal{H}_{\Omega E}$ the Hilbert space of all measurable, $\mathbb{C}^3$-valued functions defined on Ω that are square integrable with the weight $\varepsilon^{\lambda\nu}$ and the scalar product

$$(\mathbf{E}^{(1)}, \mathbf{E}^{(2)})_{\Omega E} := \int_\Omega E_\lambda^{(1)} \varepsilon^{\lambda\nu} \overline{E_\nu^{(2)}} \, \mathbf{dx}^3. \tag{2.23}$$

Moreover, we denote by $\mathcal{H}_{\Omega H}$ the Hilbert space of all measurable, $\mathbb{C}^3$-valued functions defined on Ω that are square integrable with the weight $\mu^{\lambda\nu}$ and the scalar product

$$(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})_{\Omega H} := \int_\Omega H_\lambda^{(1)} \mu^{\lambda\nu} \overline{H_\nu^{(2)}} \, \mathbf{dx}^3. \tag{2.24}$$

The Hilbert space of finite-energy fields in Ω is the direct sum

$$\mathcal{H}_\Omega := \mathcal{H}_{\Omega E} \oplus \mathcal{H}_{\Omega H}. \tag{2.25}$$

For any open set O the spaces $\mathbf{C}(O), \mathbf{C}_0(O), \mathbf{C}^n(O), \mathbf{C}_0^n(O), n = 1, 2 \ldots$ are defined as the spaces $C(O), C_0(O), C^n(O), C_0^n(O), n = 1, 2 \ldots$ but with $\mathbb{C}^6$-valued functions.

We first define a_Ω in a nice set of functions where it makes sense, which we take as $\mathbf{C}_0^1(\Omega)$. In physical terms this means that we start with the minimal assumption that Maxwell's equations are satisfied in classical sense away from the boundary of Ω. a_Ω with the domain $D(a_\Omega) := \mathbf{C}_0^1(\Omega)$ is a symmetric operator in $\mathcal{H}_\Omega$, i.e. $a_\Omega \subset a_\Omega^*$. To construct a unitary dynamics that preserves energy we have to analyze the self-adjoint extensions of a_Ω, what in physical terms means that we have to make precise in what sense Maxwell's equations are solved up to $\partial\Omega$. In other words, to construct finite-energy solutions of (2.21),

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(t) \in \mathcal{H}_\Omega,$$

with constant energy

$$(\mathbf{E}(t), \mathbf{E}(t))_{\Omega E} + (\mathbf{H}(t), \mathbf{H}(t))_{\Omega H} = (\mathbf{E}(0), \mathbf{E}(0))_{\Omega E} + (\mathbf{H}(0), \mathbf{H}(0))_{\Omega H} < \infty,$$

we have to demand that the initial finite-energy fields, $(\mathbf{E}(0), \mathbf{H}(0))^T$ belong to the domain of one of the self-adjoint extensions of a_Ω. The key issue is that a_Ω has only one self-adjoint extension, i.e. it is essentially self-adjoint. Before we state this result in a precise way in theorem 2.3 let us discuss its physical consequences. Let us denote by A_Ω the unique self-adjoint extension.

We denote by kernel A_Ω the null subspace of A_Ω, i.e.,

$$\text{kernel } A_\Omega := \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in D(A_\Omega) : A_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0 \right\},$$

and by

$$\mathcal{H}_{\Omega\perp} := (\text{kernel } A_\Omega)^\perp$$

the orthogonal complement in $\mathcal{H}_\Omega$ of kernel A_Ω. Equations (2.2) are satisfied for all times if and only if $(\mathbf{E}(0), \mathbf{H}(0))^T \in (\text{kernel } A_\Omega)^\perp$. Moreover, the unique finite-energy solutions to (2.1) and (2.2) with constant energy are constructed as follows.

We take any

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_{\Omega\perp} \cap D(A_\Omega) \tag{2.26}$$

and we obtain the finite-energy solution to Maxwell's equations (2.1) and (2.2) as

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(t) = \mathrm{e}^{-\mathrm{i}tA_\Omega} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{2.27}$$

This is the unique finite-energy solution with constant energy, and with initial value at $t = 0$ given by (2.26). Note that as $\mathrm{e}^{-\mathrm{i}tA_\Omega}\mathcal{H}_{\Omega\perp} \subset \mathcal{H}_{\Omega\perp}$ equations (2.2) are satisfied for all times if they are satisfied at $t = 0$. The unitary group $\mathrm{e}^{-\mathrm{i}tA_\Omega}$ is defined via functional calculus, but we can think of it as just the operator that gives us the unique solution. We can consider more general solutions by means of the scale of spaces associated with A_Ω, but we do not go into this direction here.

Solutions to (2.3) and (2.4), in general, do not have finite energy because they do not have enough decay at infinity to be square integrable over all Ω. Then, we only require that they are of locally finite energy in the sense that the electric and the magnetic fields are square integrable over every bounded subset of Ω, respectively, with the weight $\varepsilon^{\lambda\nu}$, and $\mu^{\lambda\mu}$. Moreover, in order that the problem (2.3) and (2.4) is well-posed—in the sense that it is self-adjoint—the solutions with locally finite energy have to be locally in the domain of the only self-adjoint extension of a_Ω, that is to say, they have to be in the domain of A_Ω when multiplied by any continuously differentiable function with support in a bounded subset of $\overline{\Omega}$. Hence, we define

Definition 2.2 (Solutions with Locally Finite Energy)**.** *We say that the fields* $(\mathbf{E},\mathbf{H})^T$ *are a solution to (2.3) and (2.4) with locally finite energy in* Ω *if they solve (2.3) and (2.4) in distribution sense in* Ω*, if, furthermore, for every bounded set* $O \subset \Omega$

$$\int_O E_\lambda \varepsilon^{\lambda\nu}\overline{E_\nu}\,\mathrm{d}\mathbf{x}^3 + \int_O H_\lambda \mu^{\lambda\nu}\overline{H_\nu}\,\mathrm{d}\mathbf{x}^3 < \infty, \tag{2.28}$$

and if for any continuously differentiable function ϕ *with bounded support in* $\overline{\Omega}$, $\phi(\mathbf{E},\mathbf{H})^T \in D(A_\Omega)$.

Similarly, given any bounded set $O \subset \Omega$ we say that the fields $(\mathbf{E},\mathbf{H})^T$ are a finite-energy solution in O if (2.28) is satisfied, if the $(\mathbf{E},\mathbf{H})^T$ are a solution to (2.3) and (2.4) in distribution sense in O and if for any continuously differentiable function ϕ with support in $\overline{O}$, $\phi(\mathbf{E},\mathbf{H})^T \in D(A_\Omega)$.

To state theorem 2.3 and to further analyze our problem we consider now Maxwell's equations in $\mathbb{R}^3_0$ and we define the Hilbert spaces of electric and magnetic fields with finite energy. The $\mathbf{E}_0, \mathbf{H}_0, \mathbf{B}_0, \mathbf{D}_0$ were defined in Ω_0, but since $\mathbb{R}^3_0\backslash\Omega_0 = \{\mathbf{c}_j\}_{j=1}^N$ is of measure zero, we can consider them as defined in $\mathbb{R}^3_0$, what we do below.

We denote by $\mathcal{H}_{0E}$ the Hilbert space of all measurable, square integrable, $\mathbb{C}^3$-valued functions defined on $\mathbb{R}^3_0$ with the scalar product

$$\left(\mathbf{E}_0^{(1)}, \mathbf{E}_0^{(2)}\right)_{0E} := \int_{\mathbb{R}^3_0} E^{(1)}_{0\lambda}\varepsilon_0^{\lambda\nu}\overline{E^{(2)}_{0\nu}}\,\mathrm{d}\mathbf{y}^3. \tag{2.29}$$

We similarly define the Hilbert space, $\mathcal{H}_{0H}$, of all measurable, square integrable, $\mathbb{C}^3$-valued functions defined on $\mathbb{R}^3_0$ with the scalar product

$$\left(\mathbf{H}_0^{(1)}, \mathbf{H}_0^{(2)}\right)_{0H} := \int_{\mathbb{R}^3_0} H^{(1)}_{0\lambda}\mu_0^{\lambda\nu}\overline{H^{(2)}_{0\nu}}\,\mathrm{d}\mathbf{y}^3. \tag{2.30}$$

The Hilbert space of finite-energy fields in $\mathbb{R}^3_0$ is the direct sum

$$\mathcal{H}_0 := \mathcal{H}_{0E} \oplus \mathcal{H}_{0H}. \tag{2.31}$$

We now write Maxwell's equations in $\mathbb{R}^3_0$ in Schrödinger form. As before we denote by ε_0 and $\boldsymbol{\mu}_0$, respectively, the matrices with entries $\varepsilon_{0\lambda\nu}$ and $\mu_{0\lambda\nu}$. By a_0 we denote the following formal differential operator:

$$a_0\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} = \mathrm{i}\begin{pmatrix}\varepsilon_0\nabla\times\mathbf{H}_0\\ -\mu_0\nabla\times\mathbf{E}_0\end{pmatrix}. \tag{2.32}$$

Then, equation (2.1) in $\mathbb{R}_0^3$ is equivalent to

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} = a_0\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix}. \tag{2.33}$$

Let us denote by $\mathbf{C}_0^\infty(\mathbb{R}_0^3)$ the set of all $\mathbf{C}^6$-valued continuously differentiable functions on $\mathbb{R}_0^3$ that have compact support in $\mathbb{R}_0^3$. Then, a_0 with the domain $\mathbf{C}_0^\infty(\mathbb{R}_0^3)$ is a symmetric operator in $\mathcal{H}_0$, i.e., $a_0 \subset a_0^*$. Moreover, it is essentially self-adjoint in $\mathcal{H}_0$, i.e., it has only one self-adjoint extension, which we denote by A_0. Its domain is given by

$$D(A_0) = \left\{\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} : a_0\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} \in \mathcal{H}_0\right\} \tag{2.34}$$

and

$$A_0\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} = a_0\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix}, \qquad \begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} \in D(A_0), \tag{2.35}$$

where the derivatives in the right-hand sides of (2.34) and (2.35) are taken in distribution sense in $\mathbb{R}_0^3$. These results follow easily from the fact that—via the Fourier transform—a_0 is unitarily equivalent to multiplication by a matrix-valued function that is symmetric with respect to the scalar product of $\mathcal{H}_0$. Moreover, it follows from explicit computation that the only eigenvalue of A_0 is zero, that it has infinite multiplicity, and that

$$\mathcal{H}_{0\perp} := (\text{kernel } A_0)^\perp = \left\{\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} \in \mathcal{H}_0 : \frac{\partial}{\partial x_\lambda}\varepsilon_0^{\lambda\nu}E_{0\nu} = 0, \frac{\partial}{\partial x_\lambda}\mu_0^{\lambda\nu}H_{0\nu} = 0\right\}. \tag{2.36}$$

Furthermore, A_0 has no singular-continuous spectrum and its absolutely-continuous spectrum is $\mathbb{R}$ (see, for example, [21, 22]).

Taking any

$$\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix} \in \mathcal{H}_{0\perp} \cap D(A_0) \tag{2.37}$$

we obtain a finite-energy solution to Maxwell's equations (2.1) and (2.2) in $\mathbb{R}_0^3$ as follows:

$$\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix}(t) = \mathrm{e}^{-\mathrm{i}tA_0}\begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix}. \tag{2.38}$$

This is the unique finite-energy solution with initial value at $t = 0$ given by (2.37). Note that as $\mathrm{e}^{-\mathrm{i}tA_0}\mathcal{H}_{0\perp} \subset \mathcal{H}_{0\perp}$ equations (2.2) are satisfied for all times if they are satisfied at $t = 0$.

We denote by U_E the following unitary operator from $\mathcal{H}_{0E}$ onto $\mathcal{H}_{\Omega E}$:

$$(U_E\mathbf{E}_0)_\lambda(\mathbf{x}) := A_\lambda^{\lambda'}E_{0\lambda'}(\mathbf{y}), \tag{2.39}$$

and by U_H the unitary operator from $\mathcal{H}_{0H}$ onto $\mathcal{H}_{\Omega H}$,

$$(U_H\mathbf{H}_0)_\lambda(\mathbf{x}) := A_\lambda^{\lambda'}H_{0\lambda'}(\mathbf{y}). \tag{2.40}$$

Then,

$$U := U_E \oplus U_H \tag{2.41}$$

is a unitary operator from $\mathcal{H}_0$ onto $\mathcal{H}_\Omega$.

We prove the following theorem in section 5.

Theorem 2.3. *The operator a_Ω is essentially self-adjoint, and its unique self-adjoint extension, A_Ω, satisfies*

$$A_\Omega = UA_0U^*. \tag{2.42}$$

Furthermore, A_Ω has no singular-continuous spectrum and its absolutely-continuous spectrum is $\mathbb{R}$. The only eigenvalue of A_Ω is zero and it has infinite multiplicity. Moreover,

$$\mathcal{H}_{\Omega\perp} := (\text{kernel } A_\Omega)^\perp = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_\Omega : \frac{\partial}{\partial x_\lambda} \varepsilon^{\lambda\nu} E_\nu = 0, \frac{\partial}{\partial x_\lambda} \mu^{\lambda\nu} H_\nu = 0 \right\}. \tag{2.43}$$

The facts that a_Ω is essentially self-adjoint and that its unique self-adjoint extension A_Ω is unitarily equivalent to the generator A_0 of the homogeneous medium are strong statements. They mean that the only possible unitary dynamics in Ω that preserves energy is given by (2.27) and that this dynamics is unitarily equivalent to the free dynamics in $\mathbb{R}^3_0$ given by (2.38). In fact, $\partial\Omega$ acts like a horizon for electromagnetic waves propagating in Ω in the sense that the dynamics is uniquely defined without any need to consider the cloaked objects $K = \cup_{j=1}^N K_j$. As we will prove below this implies, in particular, electromagnetic cloaking for all frequencies in the strong sense that the scattered wave is identically zero at each frequency and that the scattering operator in the time domain is the identity.

Formulating cloaking as a boundary value problem in Ω is of independent interest. For this purpose we introduce the following boundary condition.

Remark 2.4 (the cloaking boundary condition)**.** Let $(\mathbf{E}, \mathbf{H})^T$ be a solution with locally finite energy in Ω. According to definition 2.2 they have to be locally in the domain of A_Ω. By (2.42) this implies that

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = U \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \tag{2.44}$$

with $(\mathbf{E}_0, \mathbf{H}_0)^T$ locally in the domain of A_0, i.e., $(\mathbf{E}_0, \mathbf{H}_0)^T$ is in the domain of A_0 when multiplied by any function in $C_0^1(\mathbb{R}^3_0)$. It follows from (2.44) and (5.1), that the solutions with locally finite energy have to satisfy the cloaking boundary condition

$$\mathbf{E} \times \mathbf{n} = 0, \qquad \mathbf{H} \times \mathbf{n} = 0, \qquad \text{in} \quad \partial\Omega = \partial K_+, \tag{2.45}$$

where ∂K_+ is the outside of the boundary of the cloaked objects and $\mathbf{n}$ is the normal vector to ∂K_+.

Note that as A_Ω is the only self-adjoint extension of a_Ω, this is the only possible self-adjoint boundary condition on ∂K_+. It is self-adjoint because the matrices ε, μ are singular at ∂K_+. Then, cloaking as boundary value problem consists of finding a solution to (2.3) and (2.4) in Ω with locally finite energy that satisfies the cloaking boundary condition given in (2.45).

Let us now consider the propagation of electromagnetic waves in the cloaked objects. For this purpose we assume that in each K_j the permittivity and the permeability are given by $\varepsilon_j^{\lambda\nu}, \mu_j^{\lambda\nu}$, with inverses $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$ and where ε_j, μ_j are the matrices with entries $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$. Furthermore, we assume that $0 < \varepsilon^{\lambda\nu}(\mathbf{x}), \mu^{\lambda\nu}(\mathbf{x}) \leqslant C, \mathbf{x} \in K_j$ and that for any compact set Q contained in the interior of K_j there is a positive constant C_Q such that $\det \varepsilon^{\lambda\nu}(\mathbf{x}) > C_Q, \det \mu^{\lambda\nu}(\mathbf{x}) > C_Q, \mathbf{x} \in Q$. In other words, we only allow for possible singularities of ε_j, μ_j on the boundary of K_j.

We designate by $\mathcal{H}_{jE}$ the Hilbert space of all measurable, $\mathbb{C}^3$-valued functions defined on K_j that are square integrable with the weight $\varepsilon_j^{\lambda\nu}$ and the scalar product

$$\left(\mathbf{E}_j^{(1)}, \mathbf{E}_j^{(2)}\right)_{jE} := \int_{K_j} E_{j\lambda}^{(1)} \varepsilon_j^{\lambda\nu} \overline{E_{j\nu}^{(2)}} \, \mathrm{d}\mathbf{x}^3. \tag{2.46}$$

Similarly, we denote by $\mathcal{H}_{jH}$ the Hilbert space of all measurable, $\mathbb{C}^3$-valued functions defined on K_j that are square integrable with the weight $\mu_j^{\lambda\nu}$ and the scalar product

$$\left(\mathbf{H}_j^{(1)}, \mathbf{H}_j^{(2)}\right)_{jH} := \int_{K_j} H_{j\lambda}^{(1)} \mu_j^{\lambda\nu} \overline{H_{j\nu}^{(2)}} \, \mathrm{d}\mathbf{x}^3. \tag{2.47}$$

The Hilbert space of finite-energy fields in K_j is the direct sum

$$\mathcal{H}_j := \mathcal{H}_{jE} \oplus \mathcal{H}_{jH}, \tag{2.48}$$

and the Hilbert space in the cloaked objects K is the direct sum

$$\mathcal{H}_K := \oplus_{j=1}^N \mathcal{H}_j.$$

The complete Hilbert space of finite-energy fields including the cloaked objects is

$$\mathcal{H} := \mathcal{H}_\Omega \oplus \mathcal{H}_K. \tag{2.49}$$

We now write (2.1) as a Schrödinger equation in each K_j as before. We define the following formal differential operator:

$$a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = \mathrm{i} \begin{pmatrix} \varepsilon_j \nabla \times \mathbf{H}_j \\ -\mu_j \nabla \times \mathbf{E}_j \end{pmatrix}. \tag{2.50}$$

Equation (2.1) in K_j is equivalent to

$$\mathrm{i} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix}. \tag{2.51}$$

Let us denote the interior of K_j by $\overset{o}{K}_j := K_j \setminus \partial K_j$. Then, a_j with the domain $C_0^1(\overset{o}{K}_j)$ is a symmetric operator in $\mathcal{H}_j$. We denote

$$a := a_\Omega \oplus a_K, \qquad \text{where} \qquad a_K := \oplus_{j=1}^N a_j, \tag{2.52}$$

with the domain

$$D(a) := \left\{ \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathbf{C}_0^1(\Omega) \oplus_{j=0}^N \mathbf{C}_0^1(\overset{o}{K}_j) \right\}. \tag{2.53}$$

The operator a is symmetric in $\mathcal{H}$. The possible unitary dynamics that preserve energy for the whole system, including the cloaked objects, K, are given by the self-adjoint extensions of a. We have that

Theorem 2.5. *Every self-adjoint extension, A, of a is the direct sum of A_Ω and of some self-adjoint extension, A_K, of a_K, i.e.,*

$$A = A_\Omega \oplus A_K. \tag{2.54}$$

This theorem tells us that the cloaked objects K and the exterior Ω are completely decoupled and that we are free to choose any boundary condition inside the cloaked objects K that makes a_K self-adjoint without disturbing the cloaking effect in Ω. Boundary conditions that make A_K self-adjoint are well known (see, for example, [23–26]).

It follows from explicit computation that zero is an eigenvalue of every A_K with infinite multiplicity and that

$$\mathcal{H}_{K\perp} := (\text{kernel}\, A_K)^\perp \subset \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_K : \frac{\partial}{\partial x_\lambda} \varepsilon_K^{\lambda\nu} E_\nu = 0, \frac{\partial}{\partial x_\lambda} \mu_K^{\lambda\nu} H_\nu = 0 \right\}, \tag{2.55}$$

where we denote by $\varepsilon_K^{\lambda\nu}(\mathbf{x}) := \varepsilon_j^{\lambda\nu}(\mathbf{x})$ for $\mathbf{x} \in K_j$, and $\mu_K^{\lambda\nu}(\mathbf{x}) := \mu_j^{\lambda\nu}(\mathbf{x})$ for $\mathbf{x} \in K_j$, $j = 1, 2, \ldots, N$. It follows that zero is an eigenvalue of A with infinite multiplicity and that

$$\mathcal{H}_\perp := (\text{kernel}\, A)^\perp = \mathcal{H}_{\Omega\perp} \oplus \mathcal{H}_{K\perp}. \tag{2.56}$$

For any $\varphi = \varphi_\Omega \oplus \varphi_K \in \mathcal{H}_\perp \cap D(A)$,

$$\mathrm{e}^{-\mathrm{i}tA}\varphi = \mathrm{e}^{-\mathrm{i}tA_\Omega}\varphi_\Omega \oplus \mathrm{e}^{-\mathrm{i}tA_K}\varphi_K \tag{2.57}$$

is the unique solution of Maxwell's equations (2.1) and (2.2) with finite energy that is equal to φ at $t = 0$. This shows once again that the dynamics in Ω and in K are completely decoupled. If at $t = 0$ the electromagnetic fields are zero in Ω, they remain equal to zero for all times, and vice versa. Actually, electromagnetic waves inside the cloaked objects are not allowed to leave them, and vice versa, electromagnetic waves outside cannot go inside. In general, the solutions will be discontinuous at ∂K. This implies, in particular, that the presence of active devices inside the cloaked objects has no effect on the cloaking outside. In terms of boundary conditions this means that transmission conditions that link the electromagnetic fields inside and outside the cloaked objects are not allowed. Furthermore, choosing a particular self-adjoint extension of the electromagnetic propagator of the cloaked objects can be understood as choosing some boundary condition on the inside of the boundary of the cloaked objects. In other words, any possible unitary dynamics that preserves energy implies the existence of some self-adjoint extension in K, which can be understood as a boundary condition on the inside of the boundary of the cloaked objects. The particular self-adjoint extension, or boundary condition, that nature will take depends on the specific properties of the metamaterials used to build the transformation media as well as on the properties of the media inside the cloaked objects. Note that this does not mean that we have to put any physical surface, a lining, on the surface of the cloaked objects to enforce any particular boundary condition on the inside, since as we already mentioned this plays no role in the cloaking outside. It would be, however, of theoretical interest to see what the self-adjoint extension in K, or the interior boundary condition, turns out to be for specific cloaked objects and metamaterials. These results apply to the exact transformation media that we consider on this paper. However, the fact that there is a large class of self-adjoint extensions—or boundary conditions—that can be taken inside the cloaked objects could be useful in order to enhance cloaking in practice, where one has to consider approximate transformation media, as well as in the analysis of the stability of cloaking.

3. Scattering at fixed frequency

We now consider the scattering of plane waves by the cloaks. We have the following result.

Theorem 3.1. *For any incident plane wave at any frequency the scattered wave is identically zero.*

Proof. For simplicity, let us first consider the case where the $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are isotropic, i.e., $\varepsilon_0^{\lambda\nu} = \varepsilon^0\delta^{\lambda\nu}, \mu_0^{\lambda\nu} = \mu^0\delta^{\lambda\nu}$ and that we have an incident field that propagates along the vertical axis, x_3, with the electric field polarized along the x_1 direction,

$$\begin{pmatrix}\mathbf{E}^{\mathrm{in}}\\ \mathbf{H}^{\mathrm{in}}\end{pmatrix}(\mathbf{x}) = \begin{pmatrix}\mathbf{e}_1\,\mathrm{e}^{\mathrm{i}(kx^3-\omega t)}\\ \sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{e}_2\,\mathrm{e}^{\mathrm{i}(kx^3-\omega t)}\end{pmatrix},$$

where $k = \omega\sqrt{\varepsilon^0\mu^0}$, and $\mathbf{e}_1, \mathbf{e}_2$ are unit vectors, respectively, in the x^1 and x^2 directions.

We have to find the solution with locally finite energy to (2.3) and (2.4), $(\mathbf{E}, \mathbf{H})^T$, that outside the cloaks is equal to the sum of the incident plane wave and the scattered wave, i.e.,

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}(\mathbf{x}) = \begin{pmatrix}\mathbf{E}^{\mathrm{in}}\\ \mathbf{H}^{\mathrm{in}}\end{pmatrix}(\mathbf{x}) + \begin{pmatrix}\mathbf{E}^{\mathrm{sc}}\\ \mathbf{H}^{\mathrm{sc}}\end{pmatrix}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^3 \setminus \cup_{j=1}^N B_{b_j}(\mathbf{c}_j). \tag{3.1}$$

Furthermore, the scattered wave $(\mathbf{E}^{\mathrm{sc}}, \mathbf{H}^{\mathrm{sc}})^T$ has to satisfy the outgoing Silver–Müller radiation condition (see, for example, [27]). We compute this solution in a simple way using the unitary equivalence of A_Ω and A_0. By (2.42) the fields with locally finite energy,

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{x}) := \left(U \begin{pmatrix} \mathbf{e}_1\, \mathrm{e}^{\mathrm{i}(ky^3 - \omega t)} \\ \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \mathbf{e}_2\, \mathrm{e}^{\mathrm{i}(ky^3 - \omega t)} \end{pmatrix} \right)(\mathbf{x}),$$

are a solution to (2.3) and (2.4). Furthermore, since for $\mathbf{x} \in \mathbb{R}^3 \setminus \cup_{j=1}^N B_{b_j}(\mathbf{c}_j)$, $A^\lambda_{\lambda'} = \delta^\lambda_{\lambda'}$, we have that

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} \mathbf{e}_1\, \mathrm{e}^{\mathrm{i}(kx^3 - \omega t)} \\ \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \mathbf{e}_2\, \mathrm{e}^{\mathrm{i}(kx^3 - \omega t)} \end{pmatrix}, \qquad \mathbf{x} \in \mathbb{R}^3 \setminus \cup_{j=1}^N B_{b_j}(\mathbf{c}_j),$$

what proves that the scattered wave is identically zero.

Let us now consider the case where $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are general anisotropic media. For a discussion of plane wave solutions in this case see, for example, [21, 22, 28]. A general incident plane wave is given by

$$\begin{pmatrix} \mathbf{E}^{\mathrm{in}} \\ \mathbf{H}^{\mathrm{in}} \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} \mathbf{E}^{\mathrm{in}}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \\ \mathbf{H}^{\mathrm{in}}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \end{pmatrix},$$

where $\mathbf{k} \in \mathbb{R}^3$, $\mathbf{k} \neq 0$, and

$$(\mathbf{k} \times \mathbf{E}^{\mathrm{in}}(\mathbf{k}))^\lambda = \omega \mu^{\lambda\nu} \mathbf{H}^{\mathrm{in}}_\nu(\mathbf{k}), \qquad (\mathbf{k} \times \mathbf{H}^{\mathrm{in}}(\mathbf{k}))^\lambda = -\omega \varepsilon^{\lambda\nu} \mathbf{E}^{\mathrm{in}}_\nu(\mathbf{k}).$$

We compute again the solution with locally finite energy using the unitary equivalence between A_Ω and A_0 (2.42). We have that

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{x}) := \left(U \begin{pmatrix} \mathbf{E}^{\mathrm{in}}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{y} - \omega t)} \\ \mathbf{H}^{\mathrm{in}}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{y} - \omega t)} \end{pmatrix} \right)(\mathbf{x})$$

are a solution to (2.3) and (2.4) with locally finite energy. As in the isotropic case,

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{x}) = \begin{pmatrix} \mathbf{E}^{\mathrm{in}}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \\ \mathbf{H}^{\mathrm{in}}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \end{pmatrix}, \qquad \mathbf{x} \in \mathbb{R}^3 \setminus \cup_{j=1}^N B_{b_j}(\mathbf{c}_j),$$

what proves that the scattered wave is also zero in the general anisotropic case. □

This theorem proves that we cannot detect the cloaked objects K in any scattering experiment.

4. Scattering of wave packets

In this section, we consider the scattering of finite-energy wave packets. Of course, in practice one always sends a finite-energy wave packet on the target, which has to have a narrow enough range of frequencies in order that we can neglect dispersion, as we mentioned in the introduction.

Let χ_Ω be the characteristic function of Ω, i.e., $\chi_\Omega(\mathbf{x}) = 1, \mathbf{x} \in \Omega$, $\chi_\Omega(\mathbf{x}) = 0, \mathbf{x} \in \mathbb{R}^3 \setminus \Omega$. We define

$$\left(J \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \right)(\mathbf{x}) := \chi_\Omega(\mathbf{x}) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}(\mathbf{x}). \tag{4.1}$$

By (5.3) J is a bounded operator from $\mathcal{H}_0$ into $\mathcal{H}_\Omega$.

The wave operators are defined as follows:

$$W_{\pm} = \text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tA_{\Omega}} J \mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}, \tag{4.2}$$

provided that the strong limits exist, and where $P_{0\perp}$ denotes the projector onto $\mathcal{H}_{0\perp}$.

Let us designate by $\mathbf{W}^{1,2}(\mathbb{R}^3_0)$ the Sobolev space of $\mathbb{C}^6$-valued functions. We denote by I the identity operator on $\mathcal{H}_0$. Then,

Lemma 4.1. *The wave operators (4.2) exist and*

$$W_{\pm} = U P_{0\perp}. \tag{4.3}$$

Proof. Denote

$$W(t) := \mathrm{e}^{\mathrm{i}tA_{\Omega}} J \mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}.$$

By (2.42), for any $\varphi \in \mathcal{H}_0$

$$W(t)\varphi = \psi(t) + U P_{0\perp}\varphi, \tag{4.4}$$

with

$$\psi(t) := U \mathrm{e}^{\mathrm{i}tA_0}(U^* J - I)\, \mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}\varphi. \tag{4.5}$$

Since for $|\mathbf{y}| \geqslant R$, with R large enough, our transformation, $\mathbf{x} = f(\mathbf{y})$, is the identity, $\mathbf{x} = \mathbf{y}$, and in consequence, $A^{\lambda}_{\lambda'}(\mathbf{y}) = \delta^{\lambda}_{\lambda'}$ for $|\mathbf{y}| \geqslant R$, we have that

$$(U^* J - I) = (U^* J - I)\chi_{B_R(0)}. \tag{4.6}$$

It follows that

$$\text{s-}\lim_{t\to\pm\infty} \psi(t) = U \text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tA_0}\vartheta(t), \tag{4.7}$$

with

$$\vartheta(t) := (U^* J - I)\chi_{B_R(0)}\, \mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}\varphi. \tag{4.8}$$

We have that

$$\begin{aligned} \|\vartheta(t)\|_{\mathcal{H}_0} &\leqslant \left\| J\chi_{B_R(0)}\, \mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}} + \left\| \chi_{B_R(0)}\, \mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}_0} \\ &\leqslant C \left\| \chi_{B_R(0)}\, \mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}\varphi \right\|_{\mathcal{H}_0}. \end{aligned} \tag{4.9}$$

Then, as $(A_0 + \mathrm{i})^{-1} P_{0\perp}$ is bounded from $\mathcal{H}_0$ into $\mathbf{W}^{1,2}(\mathbb{R}^3_0)$ [21, 22], it follows from the Rellich local compactness theorem that

$$\chi_{B_R(0)}(A_0 + \mathrm{i})^{-1} P_{0\perp}$$

is a compact operator in $\mathcal{H}_0$. Suppose that $\varphi \in D(A_0) \cap \mathcal{H}_{0\perp}$. Then,

$$\text{s-}\lim_{t\to\pm\infty} \chi_{B_R(0)}\mathrm{e}^{-\mathrm{i}tA_0} P_{0\perp}\varphi = \text{s-}\lim_{t\to\pm\infty} \chi_{B_R(0)}(A_0 + \mathrm{i})^{-1} P_{0\perp}\mathrm{e}^{-\mathrm{i}tA_0}(A_0 + \mathrm{i})\varphi = 0, \tag{4.10}$$

and whence, by (4.9),

$$\text{s-}\lim_{t\to\pm\infty} \vartheta(t) = 0, \tag{4.11}$$

and it follows that in this case,

$$\text{s-}\lim_{t\to\pm\infty} \psi(t) = 0. \tag{4.12}$$

By continuity this is also true for $\varphi \in \mathcal{H}_{0\perp}$.

Then, (4.3) follows from (4.4) and (4.12). □

The scattering operator is defined as

$$S := W_+^* W_-. \tag{4.13}$$

Corollary 4.2.

$$S = P_{0\perp}. \tag{4.14}$$

Proof. This is immediate from (4.3) because $U^*U = I$. □

Let us denote by $S_\perp$ the restriction of S to $\mathcal{H}_{0\perp}$. $S_\perp$ is the physically relevant scattering operator that acts in the Hilbert space $\mathcal{H}_{0\perp}$ of finite-energy fields that satisfy equations (2.2). We designate by $I_\perp$ the identity operator on $\mathcal{H}_{0\perp}$. We have that

Corollary 4.3.

$$S_\perp = I_\perp. \tag{4.15}$$

Proof. This follows from corollary 4.2. □

The fact that $S_\perp$ is the identity operator on $\mathcal{H}_{0\perp}$ means that there is cloaking for all frequencies. Suppose that for very negative times we are given an incoming wave packet $\mathrm{e}^{-\mathrm{i}tA_0}\varphi_-$, with $\varphi_- \in \mathcal{H}_{0\perp}$. Then, for large positive times the outgoing wave packet is given by $\mathrm{e}^{-\mathrm{i}tA_0}\varphi_+$ with $\varphi_+ = S_\perp\varphi_-$. But, as $S_\perp = I$, we have that $\varphi_+ = \varphi_-$ and then

$$\mathrm{e}^{-\mathrm{i}tA_0}\varphi_- = \mathrm{e}^{-\mathrm{i}tA_0}\varphi_+.$$

Since the incoming and the outgoing wave packets are the same there is no way to detect the cloaked objects K from scattering experiments performed in Ω.

5. The proofs of theorems 2.3 and 2.5

It follows from (2.8) that the transformation matrix (2.12) is given by

$$\begin{aligned} A^\lambda_{\lambda'} &= \frac{g_j(|\mathbf{y}-\mathbf{c}_j|)}{|\mathbf{y}-\mathbf{c}_j|}\delta^\lambda_{\lambda'} + \left(\frac{g'_j(|\mathbf{y}-\mathbf{c}_j|)}{|\mathbf{y}-\mathbf{c}_j|^2} - \frac{g_j(|\mathbf{y}-\mathbf{c}_j|)}{|\mathbf{y}-\mathbf{c}_j|^3}\right)(\mathbf{y}-\mathbf{c}_j)^\lambda(\mathbf{y}-\mathbf{c}_j)^{\lambda'}, \\ &\mathbf{y} \in B_{b_j}(\mathbf{c}_j), \qquad 1 \leqslant j \leqslant N, \\ A^\lambda_{\lambda'} &= \delta^\lambda_{\lambda'} \qquad \mathbf{y} \in \mathbb{R}^3_0 \setminus \cup_{j=1}^N B_{b_j}(\mathbf{c}_j). \end{aligned} \tag{5.1}$$

The determinant is equal to

$$\begin{aligned} \Delta(\mathbf{y}) &= g'_j(|\mathbf{y}-\mathbf{c}_j|)\left(\frac{g_j(|\mathbf{y}-\mathbf{c}_j|)}{|\mathbf{y}-\mathbf{c}_j|}\right)^2, \qquad \mathbf{y} \in B_{b_j}(\mathbf{c}_j), \qquad 1 \leqslant j \leqslant N, \\ \Delta(\mathbf{y}) &= 1, \qquad \mathbf{y} \in \mathbb{R}^3_0 \setminus \cup_{j=1}^N B_{b_j}(\mathbf{c}_j). \end{aligned} \tag{5.2}$$

This result is easily obtained rotating into a coordinate system such that, $\mathbf{y}-\mathbf{c}_j = (|\mathbf{y}-\mathbf{c}_j|, 0, 0)$ [17]. Then, by (2.18) the matrices $\varepsilon^{\lambda\nu}, \mu^{\lambda\nu}$ are degenerate at ∂K and by (2.19) the matrices $\varepsilon_{\lambda\nu}, \mu_{\lambda\nu}$ are singular at ∂K. Moreover, by (2.16), (5.1) and (5.2) there is a constant C such that

$$|\varepsilon^{\lambda\nu}(\mathbf{x})| \leqslant C, \qquad |\mu^{\lambda\nu}(\mathbf{x})| \leqslant C, \qquad \mathbf{x} \in \Omega. \tag{5.3}$$

Proof of theorem 2.3. Let us denote

$\mathcal{Q} := \{(\mathbf{E}_o, \mathbf{H}_0) \in \mathcal{H}_0 : (\mathbf{E}_0, \mathbf{H}_0) = U^*(\mathbf{E}, \mathbf{H})$ for some $(\mathbf{E}, \mathbf{H}) \in C^1_0(\Omega)\}$, where U is defined in (2.41). Let us prove that $\mathcal{Q} \subset D(A_0)$. Note that $\mathcal{Q} \subset \mathbf{C}^1\left(\Omega_0 \setminus \cup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)\right)$. Suppose that $(\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{Q}$. Then,

$$a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathbf{C}\left(\Omega_0 \setminus \cup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)\right),$$

where the derivatives are defined in classical sense on $\Omega_0 \setminus \cup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)$. Furthermore, by the invariance of Maxwell's equations,

$$a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}(\mathbf{y}) = U^* a_\Omega U \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}(\mathbf{y}), \qquad \mathbf{y} \in \Omega_0 \setminus \cup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j). \tag{5.4}$$

This implies that

$$a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_0. \tag{5.5}$$

But, as $(\mathbf{E}_0, \mathbf{H}_0)$ have continuous tangential components at $\cup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)$ the function in the left-hand side of (5.4) defined for $\mathbf{y} \in \Omega_0 \setminus \cup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)$ with the derivatives in classical sense actually coincides with the distribution that is obtained when the derivatives are taken in distribution sense in $\mathbb{R}_0^3$. Then, by (5.5) we have that $(\mathbf{E}_0, \mathbf{H}_0) \in D(A_0)$, and

$$A_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}$$

with the right-hand side defined as indicated above. By (5.4) this implies that

$$A_0|_{\mathcal{Q}} = U^* a_\Omega U, \tag{5.6}$$

or what is equivalent, that

$$a_\Omega = U A_0|_{\mathcal{Q}} U^*. \tag{5.7}$$

It follows that to prove that a_Ω is essentially self-adjoint and that (2.42) holds we have to prove that A_0 is essentially self-adjoint in $\mathcal{Q}$.

In the proof below we denote by a_Ω the formal differential operator.

Suppose that $(\mathbf{E}_0, \mathbf{H}_0) \in C_0^1(\Omega_0)$. Then,

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = U^* \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \qquad \text{with} \qquad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = U \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}. \tag{5.8}$$

Hence, arguing as above, but in the opposite direction, we prove that

$$a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = U a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathcal{H}_\Omega, \tag{5.9}$$

where the function in the left-hand side with the derivatives taken in classical sense in $\Omega \setminus \cup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)$ actually coincides with the distribution obtained when the derivatives are taken in distribution sense in Ω. Let us now introduce a mollifier. Take $f \in C_0^\infty(\mathbb{R}^3)$, $f(\mathbf{x}) = 1$, $|\mathbf{x}| \leqslant 1/2$, $f(\mathbf{x}) = 0$, $|\mathbf{x}| \geqslant 1$, $\int f(x)\,\mathrm{d}x = 1$, and define $f_n(x) = n^3 f(nx)$. Denote

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_n := \int_{\mathbb{R}^3} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{x} - \mathbf{z}) f_n(\mathbf{z})\,\mathrm{d}\mathbf{z} \in C_0^\infty(\Omega), \tag{5.10}$$

for n large enough, where we have extended $(\mathbf{E}, \mathbf{H})^T$ to $\mathbb{R}^3$ by zero. Moreover,

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \text{s-}\lim_{n\to\infty} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_n, \tag{5.11}$$

and

$$\text{s-}\lim_{n\to\infty} a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_n = \text{s-}\lim_{n\to\infty} \int_{\mathbb{R}^3} a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (\mathbf{x}-\mathbf{z}) f_n(\mathbf{z})\, \mathrm{d}\mathbf{z} = a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} (\mathbf{x}). \tag{5.12}$$

The limits in (5.11) and (5.12) are in the strong convergence in $\mathcal{H}_\Omega$. It follows from (5.6), (5.8), (5.11) and (5.12) that $(\mathbf{E}_0, \mathbf{H}_0)^T$ can be approximated in the graph norm of A_0 by functions in $\mathcal{Q}$. Then, it is enough to prove that A_0 is essentially self-adjoint in $C_0^1(\Omega_0)$. But as A_0 is essentially self-adjoint in $C_0^\infty(\mathbb{R}_0^3)$ it is also essentially self-adjoint in $C_0^1(\mathbb{R}_0^3)$. Hence, it is enough to prove that any function in $C_0^1(\mathbb{R}_0^3)$ can be approximated in the graph norm of A_0 by functions in $C_0^1(\Omega_0)$. To prove this take any continuously differentiable real-valued function, f, defined on $\mathbb{R}_0^3$ such that, $f(y)=0$, $|y|\leqslant 1$ and $f(y)=1$, $|y|\geqslant 2$. Then, for any

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathbf{C}_0^1(\mathbb{R}_0^3),$$

we have that

$$\prod_{j=1}^N f(n|\mathbf{y}-\mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} \in \mathbf{C}_0^1(\Omega_0),$$

and moreover,

$$\text{s-}\lim_{n\to\infty} \prod_{j=1}^N f(n|\mathbf{y}-\mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix},$$

$$\text{s-}\lim_{n\to\infty} a_0 \prod_{j=1}^N f(n|\mathbf{y}-\mathbf{c}_j|) \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix} = a_0 \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix},$$

where by s-lim we designate the strong limit in $\mathcal{H}_0$. This completes the proof that a_Ω with the domain $C_0^1(\Omega)$ is essentially self-adjoint and that (2.42) holds.

The unitary equivalence given by (2.42) implies that A_Ω has the same spectral properties that A_0. Namely, it has no singular-continuous spectrum, the absolutely-continuous spectrum is $\mathbb{R}$ and the only eigenvalue is zero and it has infinite multiplicity. Moreover,

$$\mathcal{H}_{\Omega\perp} := (\text{kernel } A_\Omega)^\perp = \left\{ \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in \mathcal{H}_\Omega : \frac{\partial}{\partial x_\lambda} \varepsilon^{\lambda\nu} E_\nu = 0, \frac{\partial}{\partial x_\lambda} \mu^{\lambda\nu} H_\nu = 0 \right\}. \tag{5.13}$$

□

Proof of theorem 2.5. Let us denote by $\overline{a}$ the closure of a, with similar notation for a_Ω, a_j, $j=1,\ldots,N$. Then,

$$\overline{a} = A_\Omega \oplus_{j=1}^N \overline{a_j},$$

where we used the fact that as a_Ω is essentially self-adjoint, $\overline{a_\Omega} = A_\Omega$. The adjoint of a is given by

$$D(a^*) = \left\{ \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathcal{H} : \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \in D(A_\Omega), a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in \mathcal{H}_j \right\}, \tag{5.14}$$

and

$$a^* \left(\begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \right) = A_\Omega \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix}, \tag{5.15}$$

for

$$\begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in D(a^*), \tag{5.16}$$

where the derivatives are taken in distribution sense.

Let us denote by $\mathcal{K}_{\Omega\pm} := \mathrm{kernel}(\mathrm{i} \mp a_\Omega^*)$, $\mathcal{K}_{j\pm} := \mathrm{kernel}(\mathrm{i} \mp a_j^*)$ the deficiency subspaces of a_Ω and a_j, $j = 1, \ldots, N$. Since a_Ω is essentially self-adjoint $\mathcal{K}_{\Omega\pm} = \{0\}$. Let $\mathcal{K}_\pm := \oplus_{j=1}^N \mathcal{K}_{j\pm}$ be the deficiency subspaces of $a_K := \oplus_{j=1}^N a_j$. Suppose that $\mathcal{K}_\pm$ have the same dimension. Then, it follows from corollary 1 in page 141 of [19] that there is a one-to-one correspondence between self-adjoint extensions of a_K and unitary maps from $\mathcal{K}_+$ into $\mathcal{K}_-$. If V is such a unitary map, then the corresponding self-adjoint extension A_{KV} is given by

$$D(A_{KV}) = \{\varphi + \varphi_+ + V\varphi_+ : \varphi \in D(\overline{a_K}), \varphi_+ \in \mathcal{K}_+\},$$

and

$$A_{KV}\varphi = \overline{a_K}\varphi + \mathrm{i}\varphi_+ - \mathrm{i}V\varphi_+.$$

Hence, since $\mathcal{K}_{\Omega\pm} = \{0\}$ and $\overline{a} = A_\Omega \oplus \overline{a_K}$ there is a one-to-one correspondence between self-adjoint extensions of a and unitary maps, V, from $\mathcal{K}_+$ into $\mathcal{K}_-$. The self-adjoint extension A_V corresponding to V is given by

$$A_V = A_\Omega \oplus A_{KV}.$$

6. Generalizations

In this paper, we considered high-order transformation media obtained from singular transformations that blow up a finite number of points, by simplicity, and since this is the situation in the applications. Suppose that we have a transformation that is singular in a set of points that we call M and denote now $\Omega_0 := \mathbb{R}_0^3 \setminus M$. What we really used in the proofs is that $\mathbf{W}^{1,2}(\mathbb{R}_0^3) = \mathbf{W}_0^{1,2}(\Omega_0)$ where $\mathbf{W}_0^{1,2}(\Omega_0)$ denotes the completion of $\mathbf{C}_0^\infty(\Omega_0)$ in the norm of $\mathbf{W}^{1,2}(\mathbb{R}_0^3)$. We also assumed that $\varepsilon_0^{\lambda\nu}$, $\mu_0^{\lambda\nu}$ are constant. What was actually needed is that a_0 is essentially self-adjoint. Our results hold under this more general conditions provided that in (4.2), (4.3) and (4.14) we replace $P_{0\perp}$ by the projector onto the absolutely-continuous subspace of A_0 and that we assume that $D(A_0) \cap \mathcal{H}_{0ac} \subset \mathbf{W}^{1,2}(\mathbb{R}_0^3)$, where we have denoted the absolutely-continuous subspace of A_0 by $\mathcal{H}_{0ac}$. Moreover, $S_\perp$ has to be defined as the restriction of S to $\mathcal{H}_{0ac}$ and in (4.15) $I_\perp$ has to be the identity operator on $\mathcal{H}_{0ac}$. Note that under these general assumptions A_0 could have nonzero eigenvalues and singular-continuous spectrum.

For example, $\mathbf{W}^{1,2}(\mathbb{R}_0^3) = \mathbf{W}_0^{1,2}(\Omega_0)$ if M has zero Sobolev one capacity [29–31]. Moreover, assume that the permittivity and the permeability tensor densities $\varepsilon_0^{\lambda\nu}$, $\mu_0^{\lambda\nu}$ are bounded below and above. Under this condition a_0 is essentially self-adjoint. Furthermore, let us denote by $\hat{\mathcal{H}}_0$ the Hilbert space of finite-energy solutions defined as in (2.31) but with $\varepsilon_0^{\lambda\nu} = \mu_0^{\lambda\mu} = \delta^{\lambda\mu}$. Let $\hat{A}_0, \hat{\mathcal{H}}_{0\perp}$ be, respectively, the electromagnetic generator in $\hat{\mathcal{H}}_0$ and the orthogonal complement of its kernel. We have that $\mathcal{H}_0$ and $\hat{\mathcal{H}}_0$ are the same set of functions with equivalent norms. Furthermore, $D(A_0) = D(\hat{A}_0)$, kernel $\hat{A}_0 =$ kernel A_0. Moreover, $(\mathbf{E}_0, \mathbf{H}_0)^T \in \mathcal{H}_{0\perp}$ if and only if $\mathbf{E}_{0\lambda} = \sum_{\mu=0}^3 \varepsilon_{0\lambda\mu}\hat{\mathbf{E}}_{0\mu}$, $\mathbf{H}_{0\lambda} = \sum_{\mu=0}^3 \mu_{0\lambda\mu}\hat{\mathbf{H}}_{0\mu}$ for some$(\hat{\mathbf{E}}_0, \hat{\mathbf{H}}_0) \in \hat{\mathcal{H}}_{0\perp}$. As [21, 22]$D(\hat{A}_0) \cap \hat{\mathcal{H}}_{0\perp} \subset \mathbf{W}^{1,2}(\mathbb{R}_0^3)$ we have that $D(A_0) \cap \mathcal{H}_{0\perp} \subset \mathbf{W}^{1,2}(\mathbb{R}_0^3)$ if ε_0, μ_0 are bounded operators on $\mathbf{W}^{1,2}(\mathbb{R}_0^3)$ and this is true if the derivatives $\frac{\partial}{\partial y_\rho}\varepsilon_0, \frac{\partial}{\partial y_\rho}\mu_0$ are bounded operators on $\hat{\mathcal{H}}_0$ for $\rho = 1, 2, 3$. Note, furthermore, that $\mathcal{H}_{0ac} \subset \mathcal{H}_{0\perp}$.

7. Conclusion and outlook

We gave a rigorous mathematical proof, in the time and frequency domains, that first- and high-order electromagnetic invisibility cloaks actually cloak passive and active devices in a very strong sense. This puts the theory of cloaking in exact transformation media in a firm mathematical basis that will allow us, in the next step forward, to analyze the stability of cloaking in the approximate transformation media that are used in the applications.

A novel aspect of our results is that, as we prove cloaking for transformations media that are obtained from general anisotropic materials, we prove that it is possible to cloak objects contained in general anisotropic media. A very important case of anisotropic media are crystals. Suppose that we wish to cloak an object that is contained inside a general anisotropic medium, or a crystal, with permittivity and permeability tensors, respectively, $\varepsilon_0^{\lambda\mu}$ and $\mu_0^{\lambda\mu}$. To cloak the object we proceed as follows. We have to coat it with a metamaterial whose permittivity and permeability tensors are obtained using the transformation formula (2.16)—for both of them—putting in the right-hand side, respectively, the permittivity and the permeability of the general anisotropic medium, or the crystal. Our theory shows that this object with passive and active devices will be cloaked inside the general anisotropic medium, or the crystal. As we already mentioned, this is a new result, that shows that it is not necessary to transform from isotropic media. It is possible to transform from general anisotropic media. This opens the way to many interesting potential applications, not only for cloaking, but also for guiding electromagnetic waves under quite general circumstances.

Acknowledgments

This work was partially done while I was visiting the Institut für Theoretische Physik, Eidgenössische Techniche Höchschule Zurich. I thank Professors Gian Michele Graf and Jürg Fröhlich for their kind hospitality. Part of this work was done during a visit to the Department of Mathematics and Statistics of the University of Helsinki. I thank Professor Lassi Päivärinta for his kind hospitality.

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